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SOME ESTIMATES ON ENTROPY NUMBERS

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ABSTRACT

First we show that the Carl–Maurey inequality for entropy numbers

$$e_k(S) \leq c_p \left(\frac{1+\ln(n/k)}{k}\right)^{1-1/p} \|S\|$$

just characterizes weak type p spaces ($S\colon l_1^n \to X, 1 \leq k \leq n, 1 \leq p < 2$). Second we generalize a result of Gordon, König and Schütt for operators T acting between weak type 2 spaces X and weak cotype 2 spaces Y

$$(1/c(\delta))e_{[(1+\delta)n]}(T) \leq \sup_{k=1,\dots,n} 2^{-n/k}(\Pi_{i=1}^k c_i(T))^{1/k} \leq c(X,Y)e_n(T)$$

($0 < \delta \leq 1$). This estimate does not hold for $\delta = 0$.

Introduction

Using an idea of Maurey and Carl [CAR] proved the following entropy estimates for operators S acting on l_1^n into a type p space X $(1 \leq p \leq 2)$

$$e_k(S) \leq c_p \left(\frac{1+\ln(n/k)}{k}\right)^{1-1/p} \|S\|$$

for all $k = 1,\dots,n$. On the other hand, in the proportional case Pajor [PAJ] showed that X is of weak type 2 if and only if for all $S\colon l_1^n \to X$

$$e_n(S) \leq c\, n^{-1/2}\|S\|.$$

A similar result was derived by Mascioni [Ma2] for the entropy moduli and weak type $p, 1 \leq p \leq 2$. Surprisingly, we are able to show that for $1 \leq p < 2$ the proportional estimate implies the general Carl–Maurey inequality:

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THEOREM 1: *Let $1 \le p < 2, Y$ a K-convex Banach space and $T \in \mathcal{L}(X,Y)$. Then the following are equivalent:*

(1) *T is of weak type p.*

(2) *There is a constant $c_1(T) \ge 0$ such that for all $k,n \in \mathbb{N}$, $1 \le k \le n$ and $S \in \mathcal{L}(l_1^n, X)$*

$$e_k(TS) \le c_1(T)\left(\frac{1+\ln(n/k)}{k}\right)^{1-1/p} \|S\|.$$

(3) *There is a constant $c_2(T) \ge 0$ such that for all $n \in \mathbb{N}$ and $S \in \mathcal{L}(l_1^n, X)$*

$$e_n(TS) \le c_2(T) n^{1/p-1} \|S\|.$$

For the best constants we obtain

$$c_2(t) \le c_1(T) \le c_0 \frac{1}{1/p - 1/2} \omega\tau_p(T) \quad \text{and} \quad \omega\tau_p(T) \le c_0 K(Y) c_2(T).$$

For the Proof we use the same probabilistic techniques as Carl and Maurey, but observing that they are of local nature.

The second result is motivated by the result due to Gordon, König and Schütt [GKS] (Proposition 1.7) which provides an asymptotic formula for entropy numbers of a diagonal operator acting in a Banach space with an unconditional basis. In particular, for an arbitary compact operator T acting between Hilbert spaces one has

$$\frac{1}{12}\, e_n(T) \le \sup_{k=1,\dots,n} 2^{-n/\mu k} \left(\prod_{i=1}^{k} c_i(T)\right)^{1/k} \le e_n(T).$$

It is natural to ask for the corresponding relations in arbitrary Banach spaces. In particular, König and Milman conjectured that the above inequality holds for operators T acting between a type 2 space and a cotype 2 space. The following Theorem gives a complete answer to this problem.

THEOREM 2: *Let $0 < \delta \le 1$. Then there is a constant $b(\delta) > 0$ such that for all Banach spaces X of weak type 2, Y of weak cotype 2, operators $T \in \mathcal{L}(X;Y)$, $s \in \{t, c, d\}$ and $n \in \mathbb{N}$*

$$\frac{1}{b(\delta)} \max\{e_{[(1+\delta)n]}(T), e_{[(1+\delta)n]}(T^*)\} \le \sup_{k=1,\dots,n} 2^{-n/\mu k} \left(\prod_{i=1}^{k} s_i(T)\right)^{1/k}$$
$$\le c_0 \omega\tau_2(\underset{X}{\mathrm{id}}) \omega c_2(\underset{Y}{\mathrm{id}})(1 + \ln \omega c_2(\underset{Y}{\mathrm{id}})) \min\{e_n(T), e_n(T^*)\}.$$

For the best constant we obtain

$$\frac{1}{c_o}\,\frac{1}{\delta} \leq b(\delta) \leq c_0\left(1+\frac{\ln^4\delta}{\delta^3}\right).$$

In particular, Theorem 2 implies the duality of entropy numbers of an operator from X to Y, if X^* and Y are of weak cotype 2 and one of them K-convex, improving the results by Gordon, König and Schütt [GKS] and by Pajor, Tomczak-Jaegermann [PAT]. We want to point out that the lower estimate of $b(\delta)$ means that $e_{[(1+\delta)n]}(T)$ cannot be replaced by $e_n(T)$.

In section 2 we show that the first inequality of Theorem 2 holds in arbitrary Banach spaces. We learnt the ideas of the presented Proof from Carl and we are grateful to him for telling us about a rough version which he proved with Hess [CAH]. We also thank H. König for the support during the preparation of this paper.

Preliminaries

In what follows c_0 denotes always a universal constant. The parameter μ is defined to be 1 in the real case and 2 in the complex case. We use standard Banach space notations. For all Banach spaces X and subspaces $E \subset X$ we set Q_E: $X \to X/E, x \mapsto x+E, i_E$: $E \to X, x \mapsto x$. The classical spaces l_p^n, l_p, $0 < p \leq \infty$, $n \in \mathbb{N}$ are defined in the usual way. Denote by $i_{p,q}^n$: $l_p^n \to l_q^n$, $i_{p,q}$: $l_p \to l_q$ $(p \leq q)$ the formal identity map and by B_p^n the unit ball in l_p^n.

Standard references on s-numbers and operator ideals are the monographs of Pietsch [PI1] and [PI2]. The ideals of all linear bounded, finite rank operators are denoted by $\mathcal{L}$, $\mathcal{F}$, respectively.

For a Banach ideal (A,α) the component $A^*(X,Y)$ of the conjugate ideal (A^*,α^*) is the class of all operators $T \in \mathcal{L}(X,Y)$ such that

$$\alpha^*(T) := \sup\{|\mathrm{tr}TS| |\ S \in \mathcal{F}(Y,X), \alpha(S) \leq 1\} < \infty.$$

Next we recall the usual notion of some s-numbers of an operator $T \in \mathcal{L}(X,Y)$

$$a_n(T) := \inf\{\|T - S\| \mid S \in \mathcal{L}(X,Y),\ \text{rank}(S) < n\}$$
the n-th approximation number,
$$c_n(T) := \inf\{\|Ti_E\| \mid E \subset X,\ \text{codim } E < n\}$$
the n-th Gelfalnd number,
$$d_n(T) := \inf\{\|Q_F T\| \mid F \subset Y,\ \dim F < n\}$$
the n-th Kolmogoroph number,
$$t_n(T) := a_n(iTQ)$$
the n-th Tichomirov number, where

Q: $l_1(B_X) \to X$, i: $Y \to l_\infty(B_{Y^*})$ denotes the canonical metric surjection, injection, respectively. If X or Y are Hilbert spaces then $a_n(T) = a_n(T^*)$. Note if X and Y are Hilbert spaces then $a_n(T) = c_n(T) = d_n(T) = t_n(T)$ (see [PI1], [CAS]). For the Tichmirov numbers we have $t_{n+m+k-2}(RST) \leq c_n(R)\, t_m(S)\, d_k(t)$ and $t_n(T) = t_n(T^*)$ (complete symmetry, see [CAS]).

The n-th entropy number is defined by

$$\varepsilon_n(T) := \inf\{\varepsilon > 0 \mid \exists (y_k)_{k=1}^n \subset Y \colon T(B_X) \subset \bigcup_{k=1}^n (y_k + \varepsilon B_Y)\}$$

and the n-th dyadic entropy number by

$$e_n(T) := \varepsilon_{2^{n-1}}(T).$$

The s-numbers $s \in \{a, c, d, e\}$ are multiplicative, i.e.

$$s_{n+m-1}(ST) \leq s_n(S)\, s_m(T).$$

Let det: $\mathbb{K}^{n^2} \to \mathbb{K}$ be the unique determinant, then the n-th Grothendieck number is defined by

$$\Gamma_n(T) := \sup\{|\det(\langle Tx_i, y_j^* \rangle)|^{1/n} \mid (x_k)_{k=1}^n \subset B_X, (y_k^*)_{k=1}^n \subset B_{Y^*}\}.$$

The Grothendieck numbers were mainly treated by [PS1], [PS2], [GEI] and [PAT]. We mention that

$$\Gamma_n(T) = \Gamma_n(T^*) \quad \text{and} \quad \Gamma_n(T) = \sup\{\Gamma_n(Ti_E) \mid E \subset X, \dim E = n\}.$$

The volume ratio numbers were introduced to the Banach space theory by Milman and Pisier [MP2] and mascioni [MA2]; they were also studied in [PAT].

$$vr_n(T) := \sup \left\{ \left(\frac{\mathrm{vol}(Q_F T(B_X))}{\mathrm{vol}(B_{Y/F})} \right)^{1/\mu n} \mid F \subset Y \text{ and } \operatorname{codim} F = n \right\},$$

where vol denotes the usual Lebesgue measure on $\mathbb{K}^n$. Recall that $vr_n(T) \leq c_0 2^{k/\mu n} \min\{e_k(T), e_k(T^*)\}$ for all $k \in \mathbb{N}$. For H, K Hilbert spaces and $T \in \mathcal{L}(H,K)$ we have $\Gamma_n(T) = vr_n(T) = (\prod_{k=1}^n a_k(T))^{1/n}$. The multiplicativity of $s \in \{\Gamma, vr\}$ is given by $s_n(ST) \leq s_n(S)\ s_n(T)$. For operators $T \in \mathcal{L}(X,Y)$ between arbitrary Banach spaces, $n \in \mathbb{N}$ and $s \in \{t, c, d\}$ it is known by Carl [CAR] that

$$\left(\prod_{k=1}^{n} s_k(T) \right)^{1/n} \leq \Gamma_n(T). \tag{1}$$

Recall that $s \in \{c, t\}$ is injective, i.e. for each metric injection $I \in \mathcal{L}(Y,Z)$ we have $s_n(IT) = s_n(T)$, whereas e is injective up to a factor $\frac{1}{2}$, i.e. $\frac{1}{2} e_n(T) \leq e_n(IT) \leq e_n(T)$. on the other hand $s \in \{d, t, e, vr\}$ is surjective, i.e. for each metric surjection $Q \in \mathcal{L}(Z, X)$ we have $s_n(TQ) = s_n(T)$.

An operator $T \in \mathcal{L}(X,Y)$ is absolutely 2-summing ($T \in \Pi_2(X,Y)$) if there is a constant $c \geq 0$ such that for all $n \in \mathbb{N}$, $(x_k)_{k=1} \subset X$

$$\left(\sum_{k=1}^{n} \|Tx_k\|^2 \right)^{1/2} \leq c \sup_{x^* \in B_{X^*}} \left(\sum_{k=1}^{n} |\langle x_k, x^* \rangle|^2 \right)^{1/2}.$$

We denote $\pi_2(T) := \inf c$, where the infimum is taken over all c satisfying the above inequality.

Let $(g_k)_{k \in \mathbb{N}}$ be a sequence of independent, standard gaussian random variables. With this notion we define for any $n \in \mathbb{N}$ and operator $u \in \mathcal{L}(l_2^n, X)$

$$l(u) := \| \sum_{k=1}^{n} g_k u(e_k) \|_{L_2(X)}.$$

An operator $T \in \mathcal{L}(X,Y)$ is γ-summing ($T \in l(X,Y)$) if there is a constant $c \geq 0$ such that $l(Tu) \leq c\|u\|$ for all $n \in \mathbb{N}$ and $u \in \mathcal{L}(l_2^n, X)$. We denote $l(T) := \inf c$, where the infimum is taken over all $c \geq 0$ satisfying the above inequality.

As is well-known (see e.g. [PS1], [TOJ]), $l^*(v) \leq l(v^*)$ for all $v \in \mathcal{L}(X, l_2)$. But in general the conserve inequality does not hold. Hence a Banach space X is K-convex if there is a constant $c \geq 0$ such that $l(v^*) \leq c\ l^*(v)$ for all $v \in l^*(X, l_2)$. In this case we denote $K(X) := \inf c$, where the infimum is taken over all c satisfying the above inequality.

At the end of the preliminaries we mention an inequality for products of a non-increasing, positive sequence $(s_k)_{k \in \mathbb{N}}$. For this we define $[x]$ as the integer part of x for $1 \leq x < \infty$ and if $0 < x \leq 1$ we set $[x] := 1$. Then for $0 \leq \delta \leq 1$ and $n \in \mathbb{N}$

$$\left(\prod_{k=1}^{n} s_k\right)^{1/n} \leq \left(\prod_{k=1}^{[\frac{n}{(1+\delta)}]} s_{[\delta k]+k-1}\right)^{1/[\frac{n}{(1+\delta)}]} . \tag{2}$$

1. The Carl–Maurey inequality

We start with the notion of weak type operators due to Pisier [PS1] (see also [MA1]). An operator $T \in \mathcal{L}(X, Y)$ is of weak type $p, 1 \leq p \leq 2$ ($T \in \omega T_p(X, Y)$) if there is a constant $c \geq 0$ such that for all $v \in l^*(Y, l_2)$

$$\sup_{k \in \mathbb{N}} k^{1-1/p} a_k(vT) \leq c\ l^*(v).$$

We denote $\omega\tau_p(T) = \inf c$, where the infimum is taken over all c satisfying the above inequality.

We also need the gaussian-type 2 norm for n vectors. For this let $n \in \mathbb{N}$ and $T \in \mathcal{L}(X, Y)$. Then $\tau_2^n(T) := \inf c$, where the infimum is taken over all $c \geq 0$ satisfying for all $(x_k)_{k=1}^n \subset X$

$$\left\| \sum_{l=1}^{n} g_k T x_k \right\|_{L_2(Y)} \leq c \left(\sum_{k=1}^{n} \|x_k\|^2 \right)^{1/2} .$$

An operator $T \in \mathcal{L}(X, Y)$ is of gaussian-type 2 ($T \in T_2(X, Y)$) if

$$\tau_2(T) := \sup_{n \in \mathbb{N}} \tau_2^n(T) < \infty.$$

The following first Lemma is classical (see [MAP], [HOJ]). However, since it is the key of Theorem 1 we emphasize the local nature of this argument ($\tau_2^n(T)$). $\mathbb{E}$ denotes the usual expectation. Consequently the second Lemma is a slight improvement of a result of Carl [CAR], which goes back to Maurey (see [PS4]).

LEMMA 1.1: *Let $T \in \mathcal{L}(X,Y)$ and $n \in \mathbb{N}$. Then we have for all independent random variables $(X_k)_{k=1}^n \subset L_2(X)$*

$$\left(\mathbb{E}\left\|T\left(\sum_{k=1}^n X_k - \mathbb{E}X_k\right)\right\|^2\right)^{1/2} \leq \sqrt{8\pi}\tau_2^n(T)\left(\sum_{k=1}^n \mathbb{E}\|X_k\|^2\right)^{1/2}.$$

LEMMA 1.2: *Let $T \in \mathcal{L}(X,Y)$ and $k,n \in \mathbb{N}, 1 \leq k \leq n$. Then we have for all $S \in \mathcal{L}(l_1^n, X)$*

$$\varepsilon_{\binom{2\mu n+k-1}{k}}(TS) \leq \sqrt{8\pi\mu}\tau_2^k(T)k^{-1/2}\|S\|.$$

Since Lemma 1.1 holds the Proof of Lemma 1.2 can be directly taken from [CAR]. Let us point out that Lemma 1.2 is a local version of the original Lemma of Carl–Maurey. Now we are in a position to prove Theorem 1.

Proof of Theorem 1: (1) $\Rightarrow$ (2) Let us first assume that $\operatorname{rank}(T) \leq k$. Then we obtain for $v \in l^*(Y, l_2)$ (for the first inequality see [PI1])

$$\begin{aligned}\pi_2((vT)^*) &\leq 2\sum_{i=1}^k \frac{a_i(vT)}{i^{1/2}} \\ &\leq 2\sum_{i=1}^k i^{1/p-3/2}\sup_{i\in\mathbb{N}} i^{1-1/p}a_i(vT) \\ &\leq \frac{2}{1/p-1/2}k^{1/p-1/2}\omega\tau_p(T)l^*(v).\end{aligned}$$

Hence the Tomczak-Jaegermann characterization of type operators ([TOJ]) yields

$$\tau_2(T) \leq \frac{2}{1/p-1/2}k^{1/p-1/2}\omega\tau_p(T).$$

Therefore for arbitrary $T \in \omega T_p(X,Y)$ we get

$$\tau_2^k(T) \leq \frac{2}{1/p-1/2}k^{1/p-1/2}\omega\tau_p(T).$$

Applying Lemma 1.2 in the real case yields

$$\varepsilon_{\binom{2n+k-1}{k}}(TS) \leq \frac{2\sqrt{8\pi}}{1/p-1/2}\omega\tau_p(T)k^{1/p-1}\|S\|.$$

Now passing from k to $\frac{k}{20(1+\ln\, n/k)}$, as in the Proof of [CAR], we obtain assertion (2) for the dyadic entropy numbers. Similarly we prove the complex case.

The implications (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1): Let $v \in l^*(Y, l_2)$. Then the definition of the Grothendieck numbers yields that there is an operator $S \in \mathcal{L}(l_1^n, X)$ with $\|S\| \leq 1$ such that with (1)

$$a_n(vT) \leq \Gamma_n(vT) \leq 2\Gamma_n(vTS) \leq 2\Gamma_n(vTSi_{2,1}^n).$$

By Sudakov's inequality (see [PS1]) we have $e_n(v) \leq c_0 n^{-1/2} l(v^*)$. Hence

$$\begin{aligned} a_n(vT) &\leq 2^{5/2} e_{2n-1}(vTSi_{2,1}^n) \leq 2^{5/2} e_n(v) e_n(TS) \|i_{2,1}^n\| \\ &\leq 2^{5/2} c_0 c_3(T) l(v^*) n^{1/p-1} \\ &\leq 2^{5/2} c_0 c_3(T) K(Y) n^{1/p-1} l^*(v). \qquad ∎ \end{aligned}$$

Remark: (1) In Theorem 1 the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold without the assumption of K-convexity. If $X = Y$ and $T = \mathrm{id}_X$ condition (3) implies that the l_1^n's are not uniformly contained in X. Hence by Pieser's characterization [PS5] X has to be K-convex. Therefore in this case Theorem 1 holds without the assumption of K-convexity.

(2) The equivalence (1) $\Leftrightarrow$ (3) of Theorem 1 also holds in the case $p = 2$. Such a statement was proved by Pajor and Mascioni ([PAJ], [MA2]), provided that T is an identity operator on some Banach space. ∎

Statement (2) of Theorem 1 characterizes operators of weak type $p, p < 2$. In the case $p = 2$ our techniques yield

PROPOSITION 1.3: *Let $T \in \mathcal{L}(X, Y)$ be an operator of weak type 2. Then we have for all $k, n \in \mathbb{N}, 1 \leq k \leq n$ and $S \in \mathcal{L}(l_1^n, X)$*

$$(1) \quad e_k(TS) \leq c_0\ \omega\tau_2(T) \left(\frac{1 + \ln(n/k)}{k}\right)^{1/2} \left(1 + \ln\left(1 + \frac{k}{1 + \ln(n/k)}\right)\right) \|S\|.$$

$$(2) \quad e_k(TS) \leq c_0\ \omega\tau_2(T) \left(\frac{1 + \ln(n/k)}{k}\right)^{1/2} \|S\| \qquad \text{if rank}\,(TS) \leq k.$$

2. Volume and entropy estimates

In this chapter we give a refinement of an inequality due to Carl and Hess [CAH], which compares the n-th entropy numbers with S-numbers. We start with the following Theorem which generalizes the mentioned result of Gordon, König and Schütt [GKS] for the n-th entropy number provided that we have operators with values in an n-dimensional space.

THEOREM 2.1: *Let $0 < \delta \leq 1$ and $0 < \rho < \eta < \infty$. Then there are constants $c(\delta) \geq 0$ and $c(\eta, \rho) \geq 0$ such that for all $T \in \mathcal{L}(X, Y)$, $n \in \mathbb{N}$*

(i) $$vr_n(T) \leq c(\delta) \left(\prod_{k=1}^{[\frac{n}{(1+\delta)}]} t_k(T) \right)^{1/[\frac{n}{(1+\delta)}]} .$$

(ii) *If* $\dim Y \leq n$ *then* $$\frac{1}{c(\eta,\rho)} \max\{e_{[\eta n]}(T), e_{[\eta n]}(T^*)\} * (ii)$$
$$\leq \sup_{k=1,\dots,n} 2^{-\rho n/\mu k} vr_k(T) \leq c_0 \min\{e_{[\rho n]}(T), e_{[\rho n]}(T^*)\}.$$

For the best constants we obtain

$$1/(c_0)\ (1/\delta) \leq c(\delta) \leq c_0 \left(1 + \frac{\ln \delta}{\delta}\right), c(\eta, \rho) \leq c_0 \left(1 + \left(\frac{\ln(\eta - \rho)}{\eta - \rho}\right)^2\right).$$

Proof: For both assertions we may asume that $\dim X = \dim Y = n$. The Proof uses Pisier's isomorphism [PS1], [PS3]: for all $\frac{1}{2} < \alpha \leq 1$ there are isomorphisms $u \in \mathcal{L}(l_2^n, X)$ and $w \in \mathcal{L}(l_2^n, Y)$ such that

$$(*) \qquad \begin{aligned} &\sup_{k \in \mathbb{N}} k^\alpha \max\{e_k(u), e_k(u^*), d_k(u), e_k(u^{-1})\} \leq c_\alpha\, n^\alpha, \\ &\sup_{k \in \mathbb{N}} k^\alpha \max\{e_k(w), c_k(w^{-1}), e_k(w^{-1}), e_k((w^{-1})^*)\} \leq c_\alpha\, n^\alpha, \end{aligned}$$

where $c_\alpha \leq c_0(2\alpha - 1)^{-1/2}$.

(i) We deduce for $m := [n/(1+\delta)]$ with (2)

$$\begin{aligned} vr_n(T) &= vr_n(ww^{-1}Tuu^{-1}) \leq vr_n(w)\ vr_n(w^{-1}Tu)\ vr_n(u^{-1}) \\ &\leq 16 e_n(w) \left(\prod_{k=1}^{n} t_k(w^{-1}Tu) \right)^{1/n} e_n(u^{-1}) \\ &\leq 16 c_\alpha^2 \left(\prod_{k=1}^{m} t_{[\delta k]+k-1}(w^{-1}Tu) \right)^{1/m} . \end{aligned}$$

Hence the multiplicity of the Tichomirov numbers and $(*)$ imply

$$\begin{aligned} t_{[\delta k]+k-1}(w^{-1}Tu) &\leq t_k(T) c_{[\delta k/2]}(w^{-1}) d_{[\delta k/2]}(u) \\ &\leq t_k(T)(4/\delta)^{2\alpha} c_\alpha^2 (n/k)^{2\alpha}, \end{aligned}$$

and therefore $vr_n(T) \leq 16\ 16^2\ e^2\ c_\alpha^4\ \delta^{-2\alpha} (\prod_{k=1}^m t_k(T))^{1/m}$.

For $\delta \geq 1/e^2$ we obtain $c(\delta) \leq c_0$. In the case $\delta \leq 1/e^2$ we choose $\alpha = \frac{1}{2} - 1/\ln\delta \in (1/2, 1]$ and get the desired inequality with the upper estimate for $c(\delta)$.

(ii) We set $\nu := (\eta - \rho)/3$ and $\sigma := \rho + \nu$. Then the result of Gordon, König and Schütt yields

$$\begin{aligned} e_{[\eta n]}(T) &\leq e_{[\nu n]}(w)\; e_{[\sigma n]}(w^{-1}Tu)\; e_{[\nu n]}(u^{-1}) \\ &\leq c_\alpha^2 (2/\nu)^{2\alpha}\; 24 \sup_{k=1,\dots,n} 2^{-\sigma n/\mu k} vr_k(w^{-1}Tu). \end{aligned}$$

Hence the multiplicity of the volume ratio numbers and $(*)$ imply

$$\begin{aligned} 2^{-\sigma n/\mu k} vr_k(w^{-1}Tu) &\leq 2^{-\sigma n/\mu k} vr_k(w^{-1}) vr_k(T)\; vr_k(u) \\ &\leq c_\alpha^2\; 16\; (n/k)^{2\alpha}\; 2^{-(\sigma n - \rho n)/\mu k} \sup_{i=1,\dots,n} 2^{-\rho n/\mu i}\; vr_i(T) \\ &\leq c_\alpha^2\; 16\; \mu^{2\alpha} \left(\frac{2\alpha}{\sigma - \rho}\right)^{2\alpha} \sup_{i=1,\dots,n} 2^{-\rho n/\mu i} vr_i(T). \end{aligned}$$

and therefore $e_{[\eta n]}(T) \leq 384 \cdot 12^2 \cdot 6^2 \cdot c_\alpha^4\; (\eta - \rho)^{-4\alpha} \sup_{k=1,\dots,n} 2^{-\rho n/\mu k} vr_k(T)$.

For the estimate of $e_{[\eta n]}(T^*)$ we proceed in a similar way. For $\eta - \rho \geq 1/e$ we obtain then $c(\eta, \rho) \leq c_0$. In the case $\eta - \rho \leq 1/e$ we choose

$$\alpha = \frac{1}{2}\left(1 - \frac{1}{\ln(\eta - \rho)}\right) \in (1/2, 1]$$

and get the desired estimate for $c(\eta, \rho)$.

(3) For the lower estimate of $c(\delta)$ we need an appropriate operator. Choose the diagonal operator $D := \sum_{i \in \mathbb{N}} 2^{-i} e_i \otimes e_i \in \mathcal{L}(l_\infty, l_1)$. Since $a_k(D) = \sum_{i=k}^\infty 2^{-i} = 2^{1-k}$ (see [PI2]) for $k \in \mathbb{N}$ we have

$$\left(\prod_{i=1}^k a_i(D)\right)^{1/k} = 2 \cdot 2^{-(k+1)/2}.$$

For the investigation of the volume ratio numbers of D let $D^k := \sum_{i=1}^k 2^{-i} e_i \otimes e_i \in \mathcal{L}(l_\infty^k, l_1^k)$. Then a result of Schütt [SCH] implies

$$vr_k(D) \geq vr_k(D^k) = \left(\frac{\mathrm{vol}(D_\tau^k(B_\infty^k))}{\mathrm{vol}(B_1^k)}\right)^{1/\mu n}$$

$$\geq \left(\prod_{i=1}^{k} \tau_i\right)^{1/k} \frac{2}{\left\|\sum_{i=1}^{k} e_i\right\|_\infty} \frac{\left\|\sum_{i=1}^{k} e_i\right\|_1}{2e} \geq (1/e)\, k\, 2^{-(k+1)/2}.$$

Choose

$$n := [\frac{2(1+\delta)}{\delta \ln 2}]$$

then $c(\delta) \geq (1/3e^2)\,(1/\delta)$. ∎

Remark: In the case when $Y = l_2$ and X is K-convex, the non-trivial estimate of Theorem 2.1(2) is closely related to the strong inequality in [PAT], corollary 1.5. ∎

Next we prove our refined version of the Carl–Hess inequality.

THEOREM 2.2: *Let $0 < \rho < \eta < \infty$. Then there is a constant $b(\eta, \rho) \geq 0$ such that for all $T \in \mathcal{L}(X, Y)$ and $n \in \mathbb{N}$*

$$e_{[\eta n]}(T) \leq b(\eta, \rho) \sup_{k=1,\dots,n} 2^{-\rho n/\mu k} \left(\prod_{i=1}^{k} t_i(T)\right)^{1/k}.$$

For the best constant we obtain

$$b(\eta, \rho) \leq c_0 \left(2^\rho + (1 + \eta \ln^2 \eta) \left(1 + \frac{\ln^4(\eta - \rho)}{(\eta - \rho)^3}\right)\right).$$

Moreover, we have

$$1/(c_o)\,(1/\delta) \leq b(1+\delta, 1) \leq c_0(1 + \frac{\ln^4 \delta}{\delta^3}) \quad \textit{for all } 0 < \delta \leq 1.$$

Proof: We set $S = iTQ \in \mathcal{L}(l_1(B_X), l_\infty(B_{Y^*}))$. By the definition of Tichomirov numbers there is an operator $R \in \mathcal{L}(l_1(B_X), l_\infty(B_{Y^*}))$ with $\operatorname{rank}(R) < n+1$ and

$$\|R - S\| \leq 2\, a_{n+1}(S) = 2\, t_{n+1}(T) \leq 2\, 2^\rho \sup_{k=1,\dots,n} 2^{-\rho n/\mu k} \left(\prod_{i=1}^{k} t_i(T)\right)^{1/k}.$$

On the other hand, for all $1 \leq i \leq n$

$$t_i(R) \leq \|R - S\| + t_i(S) \leq 3\, t_i(T).$$

Let $F \subset l_\infty(B_{Y^*})$ with $\dim F = n$. Then there is an operator $\widehat{R} \in \mathcal{L}(l_1(B_X), F)$ such that $R = i_F \widehat{R}$. We define

$$\nu := \frac{\eta - \rho}{3}, \quad \sigma := \rho + 2\nu \quad \text{and} \quad \delta := \frac{\nu}{\rho + \nu} \in (0,1).$$

Since $\sigma < \eta$, Theorem 2.1 implies

$$e_{[\eta n]}(R) \leq e_{[\eta n]}(\widehat{R}) \leq c(\eta, \sigma) \sup_{k=1,\dots,n} 2^{-\sigma n/\mu k} vr_k(\widehat{R}).$$

Now let $1 \leq k \leq n$. We investigate two cases. If $k \geq \sigma/\nu$ then with $(1+\delta)(\rho+\nu) = \sigma$

$$\sigma n[k/(1+\delta)] \geq \sigma n(k/(1+\delta) - 1) = \rho n k + (\nu k - \sigma) n \geq \rho n k.$$

Therefore Theorem 2.1 yields with $j := [k/(1+\delta)]$

$$\begin{aligned} 2^{-\sigma n/\mu k} vr_k(\widehat{R}) &\leq c(\delta) 2^{-\rho n/\mu j} \left(\prod_{i=1}^{j} t_i(\widehat{R}) \right)^{1/j} \\ &\leq 3\ c(\delta) \sup_{m=1,\dots,n} 2^{-\rho n/\mu m} \left(\prod_{i=1}^{m} t_i(T) \right)^{1/m}. \end{aligned}$$

On the other hand if $k < \sigma/\nu$, then recall that the Banach–Mazur distance of a k-dimensional subspace to l_2^k is less than $k^{1/2}$. Hence we obtain

$$\begin{aligned} 2^{-\sigma n/\mu k} vr_k(\widehat{R}) &\leq k\ 2^{-\rho n/\mu k} \left(\prod_{i=1}^{k} t_i(\widehat{R}) \right)^{1/k} \\ &\leq 3\ \frac{\sigma}{\nu} \sup_{m=1,\dots,n} 2^{-\rho n/\mu m} \left(\prod_{i=1}^{m} t_i(T) \right)^{1/m}. \end{aligned}$$

Taking all together we get

$$\begin{aligned} e_{[\eta n]}(T) &\leq 2\ e_{[\eta n]}(S) \leq 2\ (\|S - R\| + e_{[\eta n]}(R)) \\ &\leq 2\ (2^\rho + c(\eta, \sigma) \max\{3c(\delta), 3\sigma/\nu\}) \sup_{m=1,\dots,n} 2^{-\rho n/\mu m} \left(\prod_{i=1}^{m} t_i(T) \right)^{1/m} \end{aligned}$$

Inserting the constants from Theorem 2.1 proves the inequality with the upper estimate for $b(\eta, \rho)$. For the lower estimate of $b(1+\delta, 1)$ we take the same operator as in the Theorem before. Hence

$$\sup_{k=1,\dots,n} 2^{-n/\mu k} \left(\prod_{i=1}^{k} t_i(D) \right)^{1/k} \leq \sqrt{2}\, 2^{-(2n/\mu)^{1/2}}.$$

For the investigation of the entropy numbers of D let $m := [(1+\delta)n]$. Then we obtain for $k := [(2m/\mu)^{1/2}] + 1$

$$e_m(D) \geq \frac{1}{2}\, 2^{-m/\mu k} vr(D) \geq \frac{1}{2}\, 2^{-m/\mu k}\ (1/e)\ k\ 2^{-(k+1)/2}$$
$$\geq (1/4e)(2m/\mu)^{1/2} 2^{-(2m/\mu)^{1/2}}.$$

Taking $n := [(1/\delta \ln 2)^2]$ yields the lower estimate for $b(1+\delta, 1)$. ∎

The Proof of Theorem 2.2 also yields the following inequality which is of interest for the duality problem of entropy numbers. It generalizes a result of König and Milman [KÖM] (see also [PS3]).

PROPOSITION 2.3: *For all $0 < \rho < \eta < \infty$ there is a constant $c(\eta,\rho) \geq 0$ such that for all $T \in \mathcal{L}(X,Y)$ and $n \in \mathbb{N}$*

$$e_{[\eta n]}(T) \leq c(\eta,\rho)(e_{[\rho n]}(R^*) + t_{n+1}(T)).$$

For the best constant we obtain

$$c(\eta,\rho) \leq c_0 \left(1 + \left(\frac{\ln(\eta-\rho)}{\eta-\rho}\right)^2\right).$$

Proof: We use the same notation as above. Moreover we set $G := \operatorname{kern} \widehat{R}$ and hence $\operatorname{codim} G \leq n$. Then there is a unique operator $V \in \mathcal{L}(l_1(B_X)/G, E)$ such that $R = i_E V Q_G$. Therefore Theorem 2.1 and the injectivity of the entropy numbers yield:

$$e_{[\eta n]}(R) \leq c(\eta,\rho) \sup_{k=1,\dots,n} 2^{-\rho n/\mu k} vr_k(VQ_G) \leq c_0 c(\eta,\rho)\ e_{[\rho n]}(R^*)$$
$$\leq c_0\ c(\eta,\rho)(\|S-R\| + e_{[\rho n]}(S^*)) \leq c_0\ c(\eta,\rho)(2\ t_{n+1}(T) + e_{[\rho n]}(T^*)).$$

Hence

$$e_{[\eta n]}(T) \leq 2\ e_{[\eta n]}(S) \leq 2\ (\|S-R\| + e_{[\eta n]}(R))$$
$$\leq 2\ (2\ t_{n+1}(T) + c_0\ c(\eta,\rho)(2t_{n+1}(T) + e_{[\rho n]}(T^*)))$$
$$\leq 4\ (1 + c_0\ c(\eta,\rho))(e_{[\rho n]}(T^*) + t_{n+1}(T)).$$ ∎

3. Description of entropy numbers by s-numbers

The Carl–Hess inequality of the previous chapter allows one to estimate entropy numbers by s-numbers. For a converse statement we need a relation between s-numbers and a geometric quantity of an operator. For example, such a relation is given by (1). We start with the notion of weak cotype spaces.

A Banach space X is of weak cotype 2 if there is a constant $c \geq 0$ such that for all $u \in l(l_2, X)$

$$\sup_{k \in \mathbb{N}} k^{1/2} a_k(Tu) \leq c\, l(u).$$

We denote $\omega c_q(T) :=$ inf c, where the infimum is taken over all c satisfying the above inequality. The following Bernstein–Jackson–type inequality is the key of Theorem 2. In the case $T = \mathrm{id}_X$ and $p = 2$ the statement was first proved by Mascioni [MA2]. We want to give a direct Proof, which yields optimal constants.

PROPOSITION 3.1: *Let $1 \leq p \leq 2$, Z a Banach space of weak cotype 2, $S \in L(Y, Z)$, $T \in L(X, Y)$ of weak type p. Then we have for all $n \in \mathbb{N}$*

$$n^{1/2-1/p} \Gamma_n(ST) \leq c_0\, \omega c_2(Z)(1 + \ln \omega c_2(Z)) vr_n(S^*)\, \omega t_p(T).$$

Proof: First note that by Milman and Pisier [MP2] weak cotype 2 spaces have the following geometric characterization: For all $n \in \mathbb{N}$, subspaces $E \subset Z$ with $\dim E = n$ and operators $v \in L(E, l_2^n)$

$$e_n(v) \leq c_0\, \omega c_2(Z)(1 + \ln \omega c_2(Z)) \pi_2(v)\, n^{-1/2}.$$

We may assume that $\dim X = n$ and that there are subspaces $T(X) \subset F \subset Y$, $S(F) \subset G$ with $\dim F = \dim G = n$ such that $\Gamma_n(ST) \leq \Gamma_n(S_{FG} T_{XF})$, where T_{XF}: $X \to F$, $x \mapsto Tx$, S_{FG}: $F \to G$, $y \mapsto Sy$. Hence by a result of John (see [PS1], [TOJ]) there is an isomorphism $v \in L(G, l_2^n)$ with $\|v^{-1}\| \leq 1$ and $\pi_2(v) \leq n^{1/2}$. Moreover, by the definition of the Grothendieck numbers there is an operator $u \in L(l_1^n, X)$ with $\|u\| \leq 1$ such that

$$\begin{aligned} \Gamma_n(ST) &\leq 2\, \Gamma_n(v\, S_{FG} T_{XF} u i_{2,1}^n) \\ &\leq 2\, vr_n(v^*)\, vr_n(S_{FG}{}^*)\, vr_n((T_{XF} u i_{2,1}^n)^*) \\ &\leq 8\, c_0\, e_n(v)\, vr_n(S_{FG}{}^*)\, e_n(T_{XF} u i_{2,1}^n). \end{aligned}$$

The surjectivity of the volume ratio numbers yields $vr_n(S_{FG}{}^*) \leq vr_n(S_{FG}{}^* i_G{}^*)$ $= vr_n((S i_F)^*) \leq vr_n(S^*)$. From the part (2) of the remark in section 1, or Proposition 1.3(2) and the injectivity of the entropy numbers we obtain

$$e_n(T_{XF} u i_{2,1}^n) \leq 2\, n^{1/2}\, e_n(Tu) \leq c_0\, n^{1/p-1/2}\, \omega \tau_p(T).$$

Taking all together completes the Proof. ∎

As a consequence we want now to prove Theorem 2.

Proof of Theorem 2: For the second inequality we use (1) and Proposition 3.1.

$$\sup_{k=1,\ldots,n} 2^{-n/\mu k}\left(\prod_{i=1} s_i(T)\right)^{1/k} \leq \sup_{k=1,\ldots,n} 2^{-n/\mu k}\Gamma_k(T)$$
$$\leq c_0\ \omega c_2(Y)(1+\ln \omega c_2(Y))\omega\tau_2(\underset{X}{\mathrm{id}}) \sup_{k=1,\ldots,n} 2^{-n/\mu k}\ vr_k(T^*)$$
$$\leq c_0^2\ \omega c_2(Y)(1+\ln \omega c_2(Y))\omega\tau_2(\underset{X}{\mathrm{id}}) \min\{e_n(T), e_n(T^*)\}.$$

The first inequality with the upper estimate for $b(\delta)$ follows from Theorem 2.2 and with the complete symmetry of the Tichomirov numbers.

Similarly as in Theorem 2.2 we take for the lower estimate the diagonal operator $D_\tau \in L(l_q, l_1)$, where $2 \leq q \leq \infty$ with $1/r = 1 - 1/q$ and $\tau := (2^{-i/r})_{i\in\mathbb{N}}$. For this operator we obtain $b(\delta) \geq c_0\ \delta^{-1/r}$. For $q \to \infty$ we get the desired estimate. ∎

Remark: (1) If X is a Banach space of type 2 and Y a Banach space of cotype 2 then we can replace in Theorem 2 the s-numbers $s \in \{t, c, d\}$ by the approximation numbers $s = a$. Indeed, this follows from an easy lemma due to [GKS] (using Maurey's extension property):

$$c_n(T) \leq a_n(T) \leq \tau_2(\underset{X}{\mathrm{id}})\ c_2(\underset{Y}{\mathrm{id}})\ c_n(T)$$

for all $T \in L(X, Y)$ and $n \in \mathbb{N}$.

(2) Theorem 2 implies the duality of entropy numbers of an operator $T \in L(X, Y)$, if X^* and Y are of weak cotype 2 and one of them is K-convex.

$$e_{[(1+\delta)n]}(T) \leq b(\delta)\ c(X, Y)\ e_n(T^*).$$

This is slightly more general ($\delta \to 0$) than the case considered in the paper of Pajor and Tomczak-Jaegermann [PAT].

(3) In particular, we proved in Theorem 2 that

$$\frac{1}{c(\delta, X, Y)} e_{[(1+\delta)n]}(T) \leq \sup_{k=1,\ldots,n} 2^{-n/\mu k} vr_k(T^*) \leq 2\ e_n(T).$$

This also can be considered as a generalization of the result of [GKS]. Although the above inequality holds in n-dimensional spaces (Theorem 2.1) some additional assumption is necessary in the general case. By a result of Meyer and Pajor [MEP] we have $vr_k(i^*_{p,q}) \leq c_0\ k^{1/q-1/p}$ for all $1 \leq p \leq q \leq \infty$ and $k \in \mathbb{N}$ which implies for all $n \in \mathbb{N}$

$$\sup_{k=1,\dots,n} 2^{-n/\mu k} vr_k(i^*_{p,q}) \leq c_0\ n^{1/q-1/p}.$$

On the other hand $i_{1,\infty}$ is not compact: $1/c_0 \leq e_n(i_{1,\infty}) \leq e_n(i_{p,q})$.

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